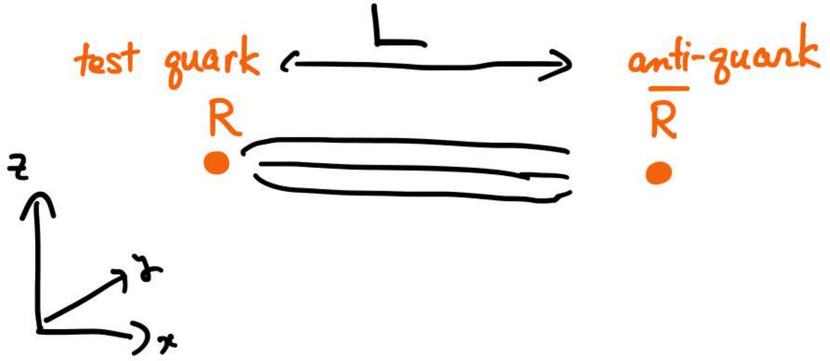
Wilson-'t Hooft classification & $\mathcal{N}=1^*$ SYM gapped vacua from the viewpoint of 1-form symmetry

Yuya Tanizaki (Yukawa institute, Kyoto)

based on 2306.02485 & w.i.p. w/ Mendel Nguyen, Mithat Ünsal (NCSU)

Confinement



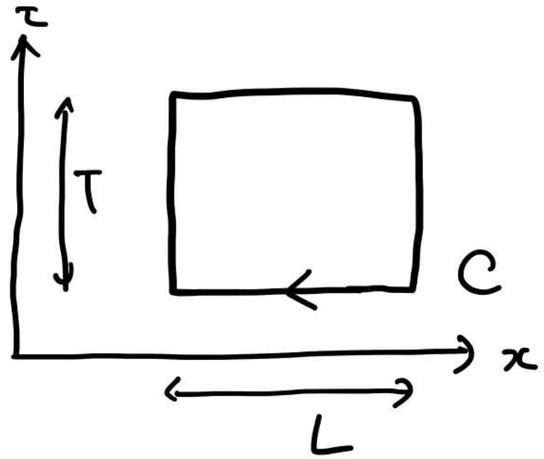
Interquark potential :

$$V_{R\bar{R}}(L) \simeq \begin{cases} \sigma_R L & \text{(confinement)} \\ \text{const.} & \text{(Higgs)} \end{cases}$$

Wilson loop : $W_R(C) = \mathrm{tr}_R \left[\mathcal{P} \exp\left(\oint_C a \right) \right]$.

$$\langle W_R(C) \rangle \simeq \exp\left(-T \cdot V_{R\bar{R}}(L) \right)$$

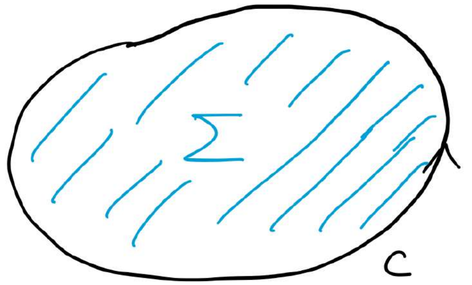
$$\simeq \begin{cases} \exp(-\sigma_R T L) & \text{(Area law)} \\ \exp(-\text{const.}\,(T+L)) & \text{(Perimeter law)}. \end{cases}$$



This is an order parameter for the $\mathbb{Z}_N$ 1-form symmetry, or center symmetry.

't Hooft pointed out the importance of magnetic lines :

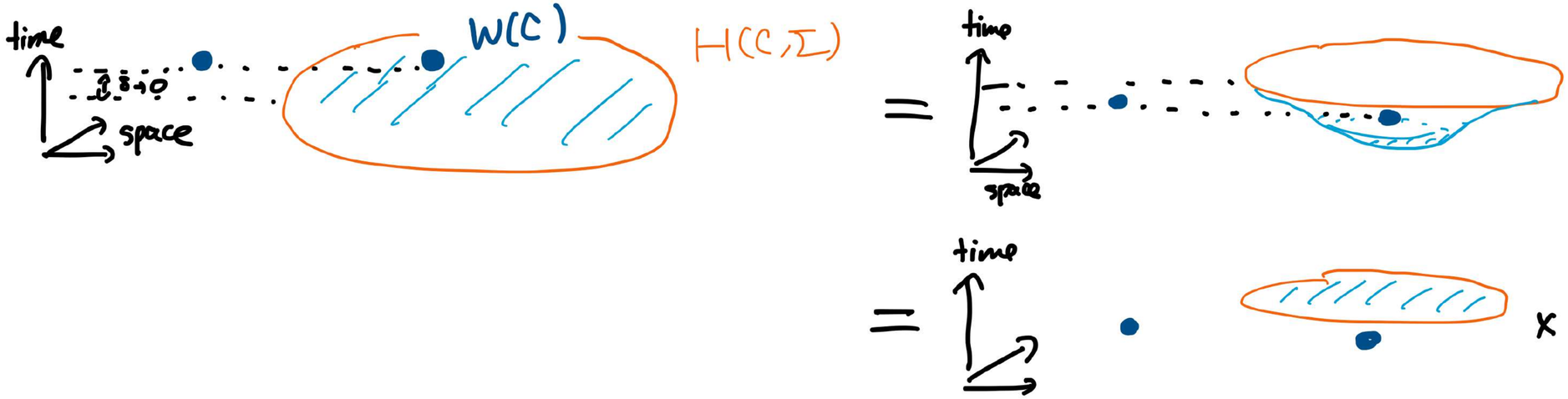
$$\begin{cases} W(C) : \text{worldline of test electric charge} \quad (R = \text{fund.}) \\ H(C,\Sigma) : \text{worldline of test magnetic charge} \quad (\text{'t Hooft loop}) \\ \qquad (\& \text{ worldsheet of associated Dirac string}) \end{cases}$$



In "SU(N)" gauge theories, the surface Σ has to be specified for $H(C,\Sigma)$.

$\Leftarrow$ Wilson-'t Hooft relation

$$\hat{W}(C)\,\hat{H}(C') = \hat{H}(C')\,\hat{W}(C)\,e^{\frac{2\pi i}{N}\,\text{Link}(C,C')}$$



Magnetic loop lives on the bdry of 1-form symmetry generator. (Aharony, Seiberg, Tachikawa '13)

Why do we introduce both loops?

To see the usefulness of having both W & H, let's see some examples for $SU(2)$:

	$W(C)$	$H(C,\Sigma)$	$W(C)H(C,\Sigma)$
Higgs	Perimeter	Area	Area
Confinement by monopoles	Area	Perimeter	Area
Confinement by dyons	Area	Area	Perimeter.

$\Rightarrow$ We get the fine classification for gapped vacua,

by specifying the charge $W_{(m,n)}(C,\Sigma) = H(C,\Sigma)^m W(C)^n$ obeying the perimeter law.

$m \in \widetilde{\mathbb{Z}}_N \times \mathbb{Z}_N$

Q. $H(C,\Sigma)$ is not a genuine line.

How do we specify "$m \in \widetilde{\mathbb{Z}}_N$" from the information of $\mathbb{Z}_N^{(1)}$ symmetry?

Main result

We show that

$$\mathbb{Z}_N^{[1]} \xrightarrow{\text{sSB}} \mathbb{Z}_{\underline{\underline{n}}}^{[1]} \quad \text{enriched with the level-} \underline{\underline{k}} \ \mathbb{Z}_{\underline{\underline{n}}}^{[1]}\text{-SPT}$$

$$\overset{1:1}{\Longleftrightarrow} \quad \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N \longrightarrow H = \{x(\underline{\underline{n}}, 0) + y(\underline{\underline{k}}, -\tfrac{N}{\underline{\underline{n}}})\} \quad \text{in Wilson-'t Hooft classification.}$$

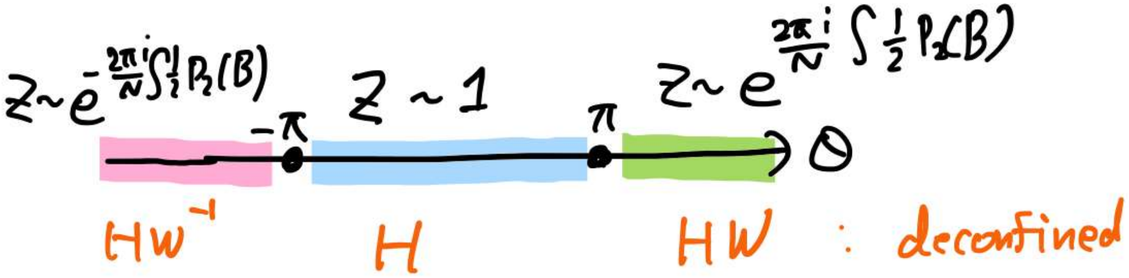
↑ Any order-N subgroup of $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ can be expressed in this way.

We can use the screening behavior of non-genuine dyonic lines to specify SPT.

(← Reminiscent of the Kennedy-Tasaki transformation of the Haldane gapped state)

example 4d pure YM + θ-term

W: always confined (at least in large-N)



(Gaiotto et al. 2017)

To get the information of symmetry, it's convenient to consider

$Z[B]$: partition function w/ background gauge field B.

- Spontaneous symmetry breaking $\mathbb{Z}_N^{(1)} \xrightarrow{SSB} \mathbb{Z}_n^{(1)}$

$$Z[B] = \underbrace{\delta(nB)}\times \cdots$$

$\int_\Sigma B$ has to be quantized in $\frac{2\pi}{n}$.

- Symmetry protected topological (SPT) state of $\mathbb{Z}_N^{(1)}$

$$Z[B] = e^{i\frac{Nk}{4\pi}\int B\wedge B} \qquad (k \sim k+N).$$

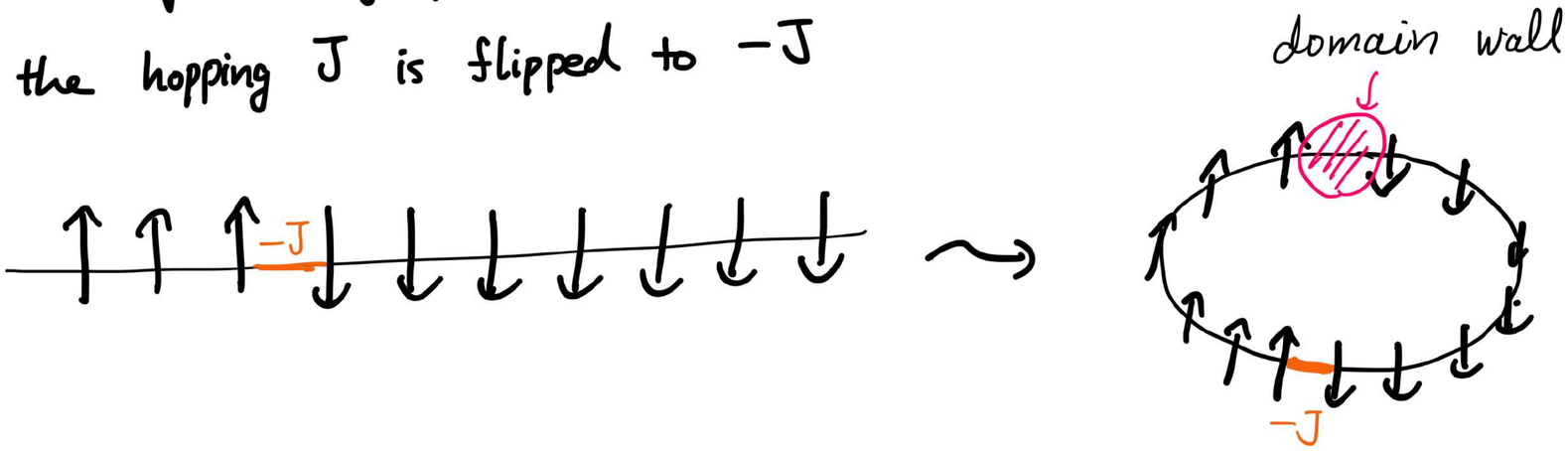
Local topological action of B.

SSB & background gauge field

Let's consider $\mathbb{Z}_2$ Ising spin for simplicity, and assume $\mathbb{Z}_2$ is spontaneously broken:

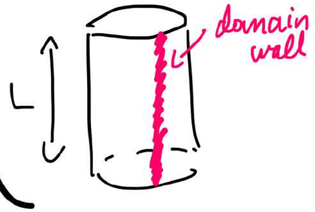
Introduce the background gauge field A w/ $\int A = \pi$.

$\Leftrightarrow$ One of the hopping J is flipped to $-J$



In the spacetime picture

$$Z[A] = \begin{cases} 1 & (A=0) \\ (\int A=\pi) = e^{-L\cdot(\text{domain-wall tension})} \to 0 \end{cases} = \delta[A].$$



With this knowledge, we can write down the low-energy effective action for

$\mathbb{Z}_N^{(1)} \xrightarrow{\text{SSB}} \mathbb{Z}_n^{(1)}$ w/ level-k $\mathbb{Z}_n^{(1)}$-SPT stacking : (Nguyen, YT, Ünsal 2306. see also Dumitrescu, et al. 2305.17084)

$$Z[B] = \delta(nB) \times e^{i \frac{nk}{4\pi} \int B \wedge B}.$$

(* If n is a nontrivial divisor of N,

$e^{i \frac{nk}{4\pi} \int B \wedge B}$ itself is not gauge invariant.

However, we can check its gauge invariance if $\delta(nB) \neq 0$.)

For this vacuum, the deconfined (non-genuine) lines are generated by

$$W_{(0,n)} = W^n \quad \& \quad W_{(\frac{N}{n},k)} = H^{N/n} W^k.$$

Our idea

We consider the S^1-compactified set up

$$M_3 \times S^1 \qquad \left(\text{size}(M_3) \gg \text{size}(S^1) \gg \frac{1}{\text{mass gap}} \right).$$

$\text{size}(M_3) \gg$: We may consider 3d effective QFTs on M_3

$\text{size}(S^1) \gg$: No phase transition under compactification

Under this compactification,

$$\left(\mathbb{Z}_N^{[1]}\right)_{4d} \Longrightarrow \left(\mathbb{Z}_N^{[1]}\right)_{3d} \times \left(\mathbb{Z}_N^{[0]}\right)_{3d}.$$

$$B_{4d} = B_m + A \wedge \frac{d\tau}{L}.$$

A: We consider the gauging of $\left(\mathbb{Z}_N^{[0]}\right)_{3d}$. (Temporal gauging)

$$\mathcal{Z}_{tH}[\tilde{B}_e, B_m] = \int \mathcal{D}a \; e^{\frac{2\pi i}{N}\int_{3d} \tilde{B}_e \cup a} \; \mathcal{Z}\left[B_m + a \wedge \frac{d\tau}{L}\right].$$

This 3d theory enjoys $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]}$ symmetry!

($\tilde{\mathbb{Z}}_N^{[1]}$: magnetic, $\mathbb{Z}_N^{[1]}$: electric)

So, we can apply the Wilson-'t Hooft scheme in a naive way.

Classification as topological states $\overset{1:1}{\Longleftrightarrow}$ Wilson-'t Hooft classification

$\mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} \mathbb{Z}_{\underline{n}}^{[1]}$ enriched with the level-k $\mathbb{Z}_{\underline{n}}^{[1]}$-SPT

$\mathbb{Z}_{N/n}$ – TQFT

Proof

$$Z_{\text{eff}}[B_{4d}] = \frac{|H^0(M_4;\mathbb{Z}_{N/n})|}{|H^1(M_4;\mathbb{Z}_{N/n})|} \sum_{[b]\in H^2(M_4;\mathbb{Z}_{N/n})} \exp\left(\frac{2\pi i}{N/n}\int b\cup B_{4d}\right)$$

$$\times \exp\left(\frac{2\pi i k}{n}\int \frac{1}{2}P_2\left(\frac{B_{4d}}{N/n}\right)\right).$$

$\Downarrow$ temporal gauging

$B_{4d} = B_m + a\wedge\frac{d\tau}{L}$

$\int \mathcal{D}a \exp\left(\frac{2\pi i}{N}\int a\cup\tilde{B}_e\right) Z[B_m + a\frac{d\tau}{L}]$

$$Z_{tH} = N^{B_1(M_3)-1}\cdot \mathcal{S}_N\left[n B_m, \frac{N}{n}\tilde{B}_e + k B_m\right]$$

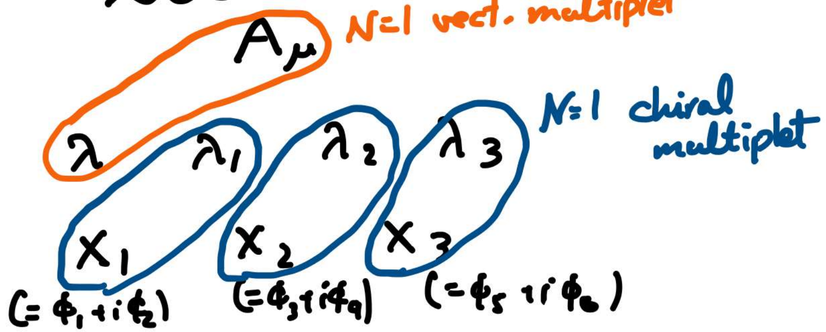
$$\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N \longrightarrow H = \{x(\underline{n},0) + y(k, -\underline{\frac{N}{n}})\}$$

(i.e. $W_{(0,n)}(C) = W^n(C)$ and $W_{(N/n,k)}(C) = H^{N/n}(C)\,W^k(C)$ are deconfined)

Application to gapped vacua of $\mathcal{N}=1^*$ SYM

Donagi, Witten '96 : Gapped vacua of $\mathcal{N}=1^*$ SYM fit into the Wilson-'t Hooft classification.

$\mathcal{N}=1^*$ SYM = $\mathcal{N}=4$ SYM + soft susy breaking to $\mathcal{N}=1$.
(= mass term to $\mathcal{N}=1$ chiral multiplets)



w/ $W(X) = \frac{1}{g^2}\mathrm{tr}\left(X_1[X_2,X_3]\right) + \frac{m}{2g^2}\mathrm{tr}\left(X_1^2+X_2^2+X_3^2\right)$

- R-sym. of $\mathcal{N}=4$ SYM is $SU(4)$.
  $U(1)$ part is explicitly broken AT THE CLASSICAL LEVEL due to $W(X)$.
  $\Rightarrow$ $\mathcal{N}=1^*$ SYM does not have $\mathbb{Z}_{2N}$ axial symmetry unlike $\mathcal{N}=1$ SYM.

Gapped vacua

Classical vacua $\frac{\partial W}{\partial X} = 0$

$$\Leftrightarrow \quad [X_i, X_j] = -m\varepsilon_{ijk} X_k$$: eq. for Lie(SU(2)) algebra

Classical config. then takes the form of

$$X_i = m \begin{pmatrix} J_i^{(n_1)} & & & \\ & J_i^{(n_2)} & & \\ & & \ddots & \\ & & & J_i^{(n_k)} \end{pmatrix}$$

w/ $J_i^{(n)}$: n-dim rep. of Lie(SU(2)) generator for

& $n_1 + \cdots + n_k = N$

In particular when

$$X_i = m \, \mathbb{1}_n \otimes J_i^{(N/n)}$$

the low-energy gauge group does not have $U(1)$'s :

$$SU(N) \xrightarrow{\text{Higgsing}} \frac{SU(n) \times \mathbb{Z}_N}{\mathbb{Z}_n} .$$

[cf. Donagi, Witten '96, Dorey '99]

As $\mathcal{N}=1$ $SU(n)$ SYM is confining,

the deconfined electric line is W^n and the theory flows to $\mathbb{Z}_{N/n}$ TQFT.

Formal S & T operations

We can apply formal S & T operations in the space of 4d QFTs with $\mathbb{Z}_N^{[1]}$ sym:

$$\begin{cases} S: \mathcal{Z}[B_{4d}] \mapsto S\cdot\mathcal{Z}[B_{4d}] = \int \mathcal{D}b_{2d}\, \mathcal{Z}[b_{4d}]\, e^{\frac{2\pi i}{N}\int b_{4d}\cup B_{4d}} \\ T: \mathcal{Z}[B_{4d}] \mapsto T\cdot\mathcal{Z}[B_{4d}] = \mathcal{Z}[B_{4d}]\, e^{\frac{2\pi i}{N}\int \frac{1}{2}P_2(B_{4d})} \end{cases}$$

Correspondingly,

electro-magnetic dual

$$S\cdot \mathcal{Z}_{tH}\left[\begin{pmatrix}\widetilde{B}_e\\ B_m\end{pmatrix}\right] = \mathcal{Z}_{tH}\left[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}\widetilde{B}_e\\ B_m\end{pmatrix}\right]$$

Witten effect

$$T\cdot \mathcal{Z}_{tH}\left[\begin{pmatrix}\widetilde{B}_e\\ B_m\end{pmatrix}\right] = \mathcal{Z}_{tH}\left[\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}\widetilde{B}_e\\ B_m\end{pmatrix}\right]$$

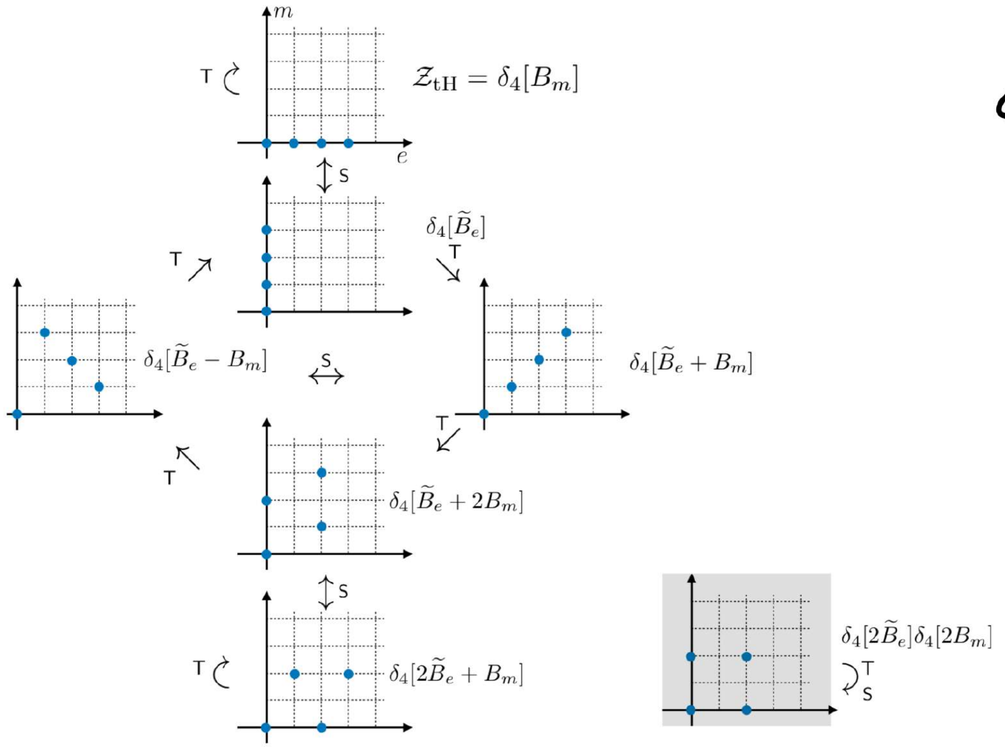
These operations form an $SL(2,\mathbb{Z}_N)$ automorphism on $\widetilde{\mathbb{Z}}_N\times\mathbb{Z}_N$, and we can relate different gapped states specified by order-N subgroups H_1, H_2 when they induce an isomorphism $H_1 \xrightarrow{\sim} H_2$.

ex. $SU(4)$ case

When $N_c=4$, the pattern of classical vacua is

Higgs vev	$SU(4) \xrightarrow{\text{Higgs}}$	$(\mathbb{Z}_4^{[1]})_{4\text{d}} \xrightarrow{\text{SSB}}$	$\mathcal{Z}_{\text{tH}}[\widetilde{B}_e, B_m]$
$J_i^{(d=4)}$	$\mathbb{Z}_4$	1	$\delta_4[B_m]$
$\mathbf{1}_2 \otimes J_i^{(d=2)}$	$[SU(2)\times\mathbb{Z}_4]/\mathbb{Z}_2$	$(\mathbb{Z}_2^{[1]})_{4\text{d}}$	$\delta_4[2B_m, 2\widetilde{B}_e + kB_m]$
$\mathbf{1}_4 \otimes J_i^{(d=1)} = 0$	$SU(4)$	$(\mathbb{Z}_4^{[1]})_{4\text{d}}$	$\delta_4[\widetilde{B}_e + kB_m]$

We note that $\mathcal{N}=1$ $SU(n)$ SYM has n vacua, and its label $k=0,1,\cdots,n-1$ appears in the low-energy partition function.



← These partition functions are consistent w/ $SL(2,\mathbb{Z}_N)$ actions.

Direct computation of $\langle W_{(n,n)}(C,\Sigma) \rangle$

We can also directly check the behaviors of loop operators in each gapped vacua.
(work in progress w/ Nguyen, Ünsal)

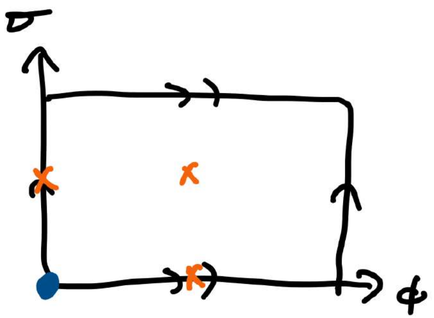
For this purpose, we consider $M_4 = \mathbb{R}^3 \times S^1$.

In this setup.

$$A_\mu \text{ in 4d} \longrightarrow \begin{cases} \phi : \text{holonomy} \\ \sigma : \text{dual photon} \end{cases}$$

and Dorey '99 proposed its nonperturbative superpotential (elliptic superpotential)

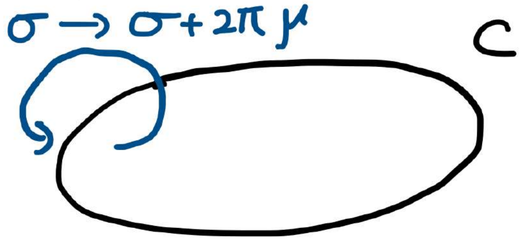
SU(2) case



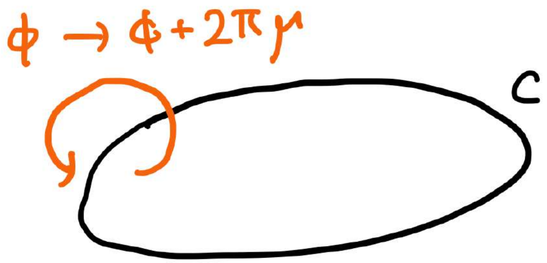
x : $\frac{\partial W}{\partial(\sigma,\phi)} = 0$

• : pole singularity

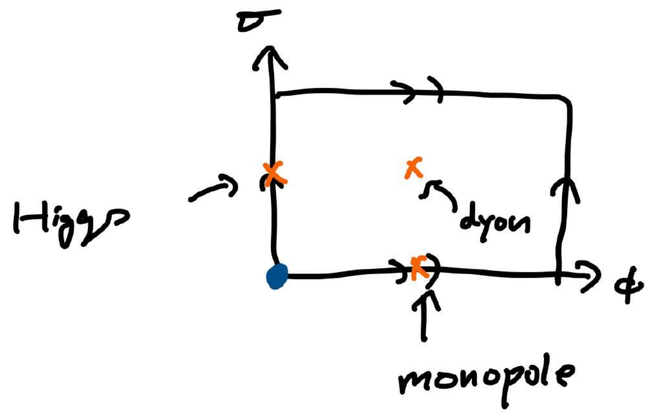
Wilson loop $W(C)$:



't Hooft loop $H(C,\Sigma)$



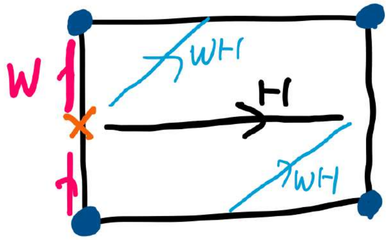
SU(2) case



× : $\frac{\partial W}{\partial(\sigma,\phi)} = 0$

● : pole singularity

Then, let's take $(\phi,\sigma) = (0,\pi)$ vacuum ($\leftrightarrow$ Higgs vacuum)



For W, the trajectory hits the singularity and it bounces back.

$\rightsquigarrow \langle W(C) \rangle$: perimeter law.

while $\langle H \rangle$, $\langle HW \rangle$: area law.

In this way, we can explicitly check the consistency of the behaviors of the loop op.

Summary

- We relate two classification schemes for gapped phases of 4d QFTs w/ $\mathbb{Z}_N^{[1]}$ sym.

$$\mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} \mathbb{Z}_{\underline{n}}^{[1]} \quad \text{enriched with the level-} \textcircled{k} \ \mathbb{Z}_{\underline{n}}^{[1]}\text{-SPT}$$

$$\overset{1:1}{\Longleftrightarrow} \quad \widetilde{\mathbb{Z}}_N \times \mathbb{Z}_N \longrightarrow H = \left\{ x\left(\underline{n}, 0\right) + y\left(\textcircled{k}, -\tfrac{N}{\underline{n}}\right) \right\} \quad \text{in Wilson-'t Hooft classification.}$$

$$Z_{\text{eff}}[B_{4d}] = \frac{|H^0(M_4; \mathbb{Z}_{N/n})|}{|H^1(M_4; \mathbb{Z}_{N/n})|} \sum_{[b] \in H^2(M_4; \mathbb{Z}_{N/n})} \exp\left(\frac{2\pi i}{N/n} \int b \cup B_{4d}\right)$$

$$\times \exp\left(\frac{2\pi i k}{n} \int \frac{1}{2} P_2\left(\frac{B_{4d}}{N/n}\right)\right).$$

$$\Big\Downarrow \quad \text{temporal gauging}$$

$$B_{4d} = B_m + a \wedge \frac{d\tau}{\mathbb{L}}$$

$$\int \mathcal{D}a \exp\left(\tfrac{2\pi i}{N} \int a \cup \widetilde{B}_e\right) Z[B_m + a\tfrac{d\tau}{\mathbb{L}}]$$

$$Z_{tH} = N^{B_1(M_3)-1} \cdot \mathcal{S}_N[n B_m, \tfrac{N}{n} \widetilde{B}_e + k B_m]$$

- We consider the formal S & T operations in the space of 4d QFTs.

- We apply these results to 4d $\mathcal{N}=1^*$ SYM theory.